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Fundamental formulation for frictional contact problems of coated systems

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Abstract

The plane problem about surface loading of an elastic layer perfectly bonded to an elastically dissimilar half-plane is considered. The fundamental solutions for concentrated forces acting perpendicular and parallel to the layer surface are obtained. The stress and displacement fields in the coating layer and the substrate due to these concentrated forces are found. On the basis of these expressions, the fundamental integral equations are obtained which describe the frictional contact between an elastic body and a coated substrate. From the fundamental integral equations, a series of integral equations for special cases are deduced corresponding to practical contact situations. Finally, the numerical results for a typical example are given to demonstrate the validity of the fundamental equations and numerical procedures given in this paper.

Keywords: Contact mechanics; Coating; Singular integral equation

1. Introduction

Great many situations in engineering require the transmission of loads through contacts between different components and parts of assemblies. Often the contacting surfaces must be allowed to undergo relative sliding with respect to each other, for example, in ball or journal bearing assemblies, or in the dovetail connection between the fan blades and root disks in aero-engines. In order to impart superior strength and durability to the assembly the contacting surfaces are often coated with materials possessing mechanical properties that are distinctly different from those of the substrate, e.g. much increased hardness and stiffness, or lower stiffness and low coefficient of friction.

Contact between coated bodies gives rise to complex stress states in the coating layers and substrates, affected by the elastic properties of the coating, substrate and indenter, the coefficient of friction, the type of contact (complete or incomplete), and the extent of contact in comparison with the coating thickness.

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Two main approaches to this class of problems have been used. One is the finite element method (FEM), employed by many researchers (Ihara et al., 1986a,b; Komovopoulus, 1988; Tian and Saka, 1991; Anderson and Collins, 1995; Lovell, 1998; Aslantas and Tasgetiren, 2002; and others). The other method is the boundary integral method (BIM), also widely investigated and used for this type of problems (e.g. Wu and Chiu, 1967; Bentall and Johnson, 1986; Gupta and Walowit, 1974; King and O'Sullivian, 1987; Jaffar and Savage, 1988; Oliveira and Bower, 1996; Elsharkawy, 1999; Porter and Hills, 2002; and others). FEM can be effectively used for arbitrary complex geometry and complex material constitutive laws, but it requires some significant effort in pre- and post-processing of the data. BIM is more convenient and straightforward than FEM when used for simplified contact configurations and linear elastic material response. For material selection and preliminary design BIM may be much more efficient than FEM. However, a complete systemic framework is still lacking for calculating contact tractions and stress fields around a contact between an elastic punch and a coated surface, in a way that is perhaps similar to contact mechanics of uncoated systems (Johnson, 1985; Hills et al., 1993), although some special simplified models have been investigated (Nowell and Hills, 1988).

The aim of the present study is to develop a general framework for calculating contact tractions and stress fields around a contact between an elastic punch and a coated surface. Various contact models (i.e. different combinations of coating, substrate, and indenter, as well as all kinds of frictional contact models, such as full stick, partial slip, and full slip models) can be derived with some additional assumptions in the BIM framework.

The solution procedure is constructed in the following steps. Firstly, fundamental solutions for concentrated normal and tangential forces acting at the surface of a coated half-plane are determined in Section 2. Solutions for the Airy stress functions for the two cases are obtained, and are used to derive the expressions for the elastic fields everywhere in the coating and substrate. Of particular interest in contact mechanics problems are the values at the coating surface, i.e. the surface tractions and displacements. The stress and displacement fields show singular behaviour when the point at which they are evaluated approaches the loading point. In Section 3, on the basis of the fundamental solutions for concentrated forces we develop fundamental singular integral equation formulation for the unknown traction distributions, for the general case of frictional contact between an elastic punch and a coated substrate. Subsequently in Section 4, some degenerate fundamental integral equations corresponding to typical contacts of coated system are derived, which are quite often encountered in practical calculations. Finally, full displacement and stress fields of coated system due to surface tractions are presented in simple forms in Section 5.

2. Fundamental solutions

The contact problem of an elastic coating of uniform thickness perfectly bonded to an underlying dissimilar elastic half-plane is investigated on the basis of the two-dimensional theory of elasticity.

Under the conditions of plane strain, the stress components σ_{xx}, σ_{yy}, and σ_{xy} can be expressed as

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}, \qquad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}, \qquad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}. \tag{2.1}$$

Here ϕ is the Airy stress function. It is usually required that function ϕ be chosen so as to satisfy only one compatibility equation relating the in-plane strain components ε_{xx}, ε_{yy}, and ε_{xy}. This leads to the requirement that ϕ be biharmonic, $\nabla^4\phi = 0$. Solution of the equation $\nabla^4\phi = 0$ can be chosen in the form

$$\phi = [(A_1 + A_2 y)e^{-wy} + (A_3 + A_4 y)e^{wy}](c_1 \cos wx + c_2 \sin wx), \tag{2.2}$$

where A_1, A_2, A_3, A_4, c_1, c_2 are parameters which are only related to parameter w.

Introduce two harmonic functions Q_1 and Q_2, where Q_1 satisfies

$$Q_1 = \nabla^2\phi = \sigma_{xx} + \sigma_{yy} \tag{2.3}$$

and Q_2 is the conjugate harmonic of Q_1, i.e. is related to Q_1 by the Cauchy–Riemann equations as follows:

$$\frac{\partial Q_1}{\partial x} = \frac{\partial Q_2}{\partial y}, \qquad \frac{\partial Q_1}{\partial y} = -\frac{\partial Q_2}{\partial x}. \tag{2.4}$$

The integral of the analytic function $f(z) = Q_1 + \mathrm{i}Q_2$ is another analytic function, $\psi(z)$,

$$\psi(z) = q_1 + \mathrm{i}q_2 = \frac{1}{c}\int f(z)\,\mathrm{d}z, \tag{2.5}$$

where c is as yet an arbitrary constant, $z = x + \mathrm{i}y$ and $\mathrm{i} = \sqrt{(-1)}$. Functions (q_1, q_2) are also conjugate harmonic functions satisfying the Cauchy–Riemann equations,

$$\frac{\partial q_1}{\partial x} + \mathrm{i}\frac{\partial q_1}{\partial x} = \frac{1}{c}(Q_1 + \mathrm{i}Q_2). \tag{2.6}$$

From Eq. (2.3) and the previous equation one can solve

$$\phi = xq_1 + yq_2 + p_0, \tag{2.7}$$

where p_0 is arbitrary function which satisfies $\nabla^2 p_0 = 0$, provided $c = 4$.

Plane strain elastic equations

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1+v}{E}[\sigma_{xx} - v(\sigma_{xx} + \sigma_{yy})],\\
\varepsilon_{yy} &= \frac{1+v}{E}[\sigma_{y} - v(\sigma_{xx} + \sigma_{yy})],\\
\varepsilon_{xy} &= \frac{1+v}{E}\sigma_{xy}
\end{aligned} \tag{2.8}$$

can be expressed in terms of ϕ, q_1 and q_2 by virtue of Eqs. (2.1), (2.3), (2.5) and (2.6) as

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1+v}{E}\left[4(1-v)\frac{\partial q_1}{\partial x} - \frac{\partial^2\phi}{\partial x^2}\right],\\
\frac{\partial v}{\partial y} &= \frac{1+v}{E}\left[4(1-v)\frac{\partial q_2}{\partial y} - \frac{\partial^2\phi}{\partial y^2}\right].
\end{aligned} \tag{2.9}$$

Integration of Eq. (2.9) gives

$$\begin{aligned}
u &= \frac{1+v}{E}\left[4(1-v)q_1 - \frac{\partial\phi}{\partial x}\right] + f_1(y),\\
v &= \frac{1+v}{E}\left[4(1-v)q_2 - \frac{\partial\phi}{\partial y}\right] + f_2(x),
\end{aligned} \tag{2.10}$$

where $f_1(y)$, $f_2(x)$ are arbitrary functions of integration. These can be found from

$$\sigma_{xy} = \frac{E}{2(1+v)}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] = -\frac{\partial^2\phi}{\partial x\partial y} + \frac{1}{2}\left(\frac{\mathrm{d}f_1}{\mathrm{d}y} + \frac{\mathrm{d}f_2}{\mathrm{d}x}\right) \tag{2.11}$$

in comparison with the third equation in Eq. (2.1). Obviously, it is necessary that $\left(\frac{\mathrm{d}f_1}{\mathrm{d}y} + \frac{\mathrm{d}f_2}{\mathrm{d}x}\right) = 0$, and hence $f_1(y)$, $f_2(x)$ are linear in their variables. Therefore they represent rigid-body rotation and can often be discarded in Eq. (2.10).

The plane strain general solution is written as

$$u = \frac{1+v}{E}\left[4(1-v)q_1 - \frac{\partial\phi}{\partial x}\right],$$
$$v = \frac{1+v}{E}\left[4(1-v)q_2 - \frac{\partial\phi}{\partial y}\right], \qquad (2.12)$$
$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}, \qquad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}, \qquad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}.$$

2.1. Fundamental solution for a concentrated load normal to the surface

The problem has been treated approximately by Gupta and Walowit (1974) when they investigated the frictionless contact problem by using Fourier transform method. Here, in order to provide a basis for several fundamental solutions of this problem in the following derivation, we develop full solutions in some detail. In view of the symmetry of the normal load (Fig. 1) and the requirement that the substrate be stress-free at large distances from the loading point, Airy stress functions for coating and substrate can be written as

$$\phi^{\mathrm{I}} = \int_0^\infty [(A_1 + A_2 y)\mathrm{e}^{-wy} + (A_3 + A_4 y)\mathrm{e}^{wy}]\cos wx\,\mathrm{d}w, \qquad (2.13)$$

$$\phi^{\mathrm{II}} = \int_0^\infty (A_5 + A_6 y)\mathrm{e}^{-wy}\cos wx\,\mathrm{d}w, \qquad (2.14)$$

where A_i are generally functions, as yet to be determined, of the dummy Fourier transform variable w; superscripts I and II refer to the coating and substrate, respectively.

Using the procedure described in the previous section, the expressions for the auxiliary functions q_1 and q_2 are obtained as follows:

$$q_1^{\mathrm{I}} = \frac{1}{2}\int_0^\infty (A_4\,\mathrm{e}^{wy} - A_2\,\mathrm{e}^{-wy})\sin wx\,\mathrm{d}w,$$
$$q_2^{\mathrm{I}} = \frac{1}{2}\int_0^\infty (A_4\,\mathrm{e}^{wy} + A_2\,\mathrm{e}^{-wy})\cos wx\,\mathrm{d}w, \qquad (2.15)$$

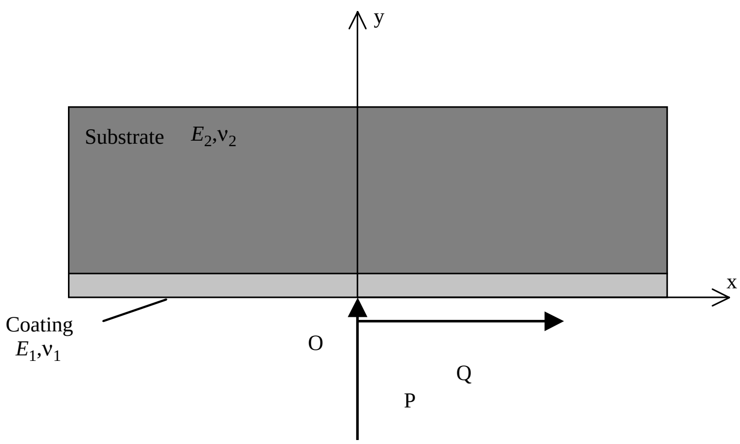

Fig. 1. Coating–substrate system subject to concentrated loads at origin.

$$q_1^{\mathrm{II}} = -\frac{1}{2}\int_0^\infty A_6\,\mathrm{e}^{-wy}\sin wx\,\mathrm{d}w,$$
$$q_2^{\mathrm{II}} = \frac{1}{2}\int_0^\infty A_6\,\mathrm{e}^{-wy}\cos wx\,\mathrm{d}w. \tag{2.16}$$

Substituting Eqs. (2.13)–(2.16) into Eq. (2.12), the general solution for the elastic fields in the coating and substrate is obtained.

The unknown functions $A_i(w)$ are now determined from the boundary conditions at the coating surface

$$\sigma_{yy}^{\mathrm{I}} = -\delta(x), \qquad \sigma_{xy}^{\mathrm{I}} = 0 \qquad (y=0) \tag{2.17}$$

and the continuity conditions of stresses and displacements along the interface

$$\sigma_{yy}^{\mathrm{I}} = \sigma_{yy}^{\mathrm{II}}, \quad \sigma_{xy}^{\mathrm{I}} = \sigma_{xy}^{\mathrm{II}} \quad (y=h), \quad u^{\mathrm{I}} = u^{\mathrm{II}}, \quad v^{\mathrm{I}} = v^{\mathrm{II}}. \tag{2.18}$$

Functions $A_i(w)$ contain the coating thickness h as a parameter, and also depend on the elastic bi-material constants for the layer and substrate, given by $B_i = 4(1+v_i)(1-v_i)/E_i$, $C = C_1 - C_2$, $C_i = (1+v_i)/E_i$, where 1 refers to the coating and 2 refers to the substrate ($i = 1, 2$). The expressions for functions $A_i(w)$ are given in Appendix A.

The stress and displacement fields for the coating and substrate can now be given in the following form:

$$d_1^{\mathrm{NI}}(x,y) = u^{\mathrm{NI}} = \frac{1}{2}\int_0^\infty \mathrm{e}^{-wy} D_1^{\mathrm{NI}}\frac{\sin wx}{w}\,\mathrm{d}w,$$
$$d_2^{\mathrm{NI}}(x,y) = v^{\mathrm{NI}} = \frac{1}{2}\int_0^\infty \mathrm{e}^{-wy} D_2^{\mathrm{NI}}\frac{\cos wx}{w}\,\mathrm{d}w,$$
$$s_{22}^{\mathrm{NI}}(x,y) = \sigma_{yy}^{\mathrm{NI}} = -\int_0^\infty \left[\mathrm{e}^{-wy}(A_1^{\mathrm{N}} + A_2^{\mathrm{N}}wy) + \mathrm{e}^{wy}(A_3^{\mathrm{N}} + A_4^{\mathrm{N}}wy)\right]\cos wx\,\mathrm{d}w, \tag{2.19}$$
$$s_{12}^{\mathrm{NI}}(x,y) = \sigma_{xy}^{\mathrm{NI}} = \int_0^\infty \mathrm{e}^{-wy}\left[(1-wy)A_2^{\mathrm{N}} - (A_1^{\mathrm{N}} - A_3^{\mathrm{N}}\mathrm{e}^{2wy}) + A_4^{\mathrm{N}}\mathrm{e}^{2wy}(1+wy)\right]\sin x\,\mathrm{d}w,$$
$$s_{11}^{\mathrm{NI}}(x,y) = \sigma_{xx}^{\mathrm{NI}} = \int_0^\infty \mathrm{e}^{-wy}\left[(A_1^{\mathrm{N}} + A_3^{\mathrm{N}}\mathrm{e}^{2wy}) + A_2^{\mathrm{N}}(wy-2) + A_4^{\mathrm{N}}\mathrm{e}^{2wy}(2+wy)\right]\cos wx\,\mathrm{d}w,$$

where

$$D_1^{\mathrm{NI}} = -A_2^{\mathrm{N}}B_1 + 2A_1^{\mathrm{N}}C_1 + 2A_2^{\mathrm{N}}C_1wy + \mathrm{e}^{2wy}(A_4^{\mathrm{N}}B_1 + 2A_3^{\mathrm{N}}C_1 + 2A_4^{\mathrm{N}}C_1wy),$$

$$D_2^{\mathrm{NI}} = A_2^{\mathrm{N}}B_1 + 2A_1^{\mathrm{N}}C_1 + 2A_2^{\mathrm{N}}C_1(wy-1) + \mathrm{e}^{2wy}\left[A_4^{\mathrm{N}}B_1 - 2A_3^{\mathrm{N}}C_1 - 2A_4^{\mathrm{N}}C_1(wy+1)\right].$$

$$d_1^{\mathrm{NII}}(x,y) = u^{\mathrm{NII}} = \frac{1}{2}\int_0^\infty \mathrm{e}^{-wy}\left(-A_6^{\mathrm{N}}B_2 + 2A_5^{\mathrm{N}}C_2 + 2A_6^{\mathrm{N}}C_2wy\right)\frac{\sin wx}{w}\,\mathrm{d}w,$$
$$d_2^{\mathrm{NII}}(x,y) = v^{\mathrm{NII}} = \frac{1}{2}\int_0^\infty \mathrm{e}^{-wy}\left[2A_5^{\mathrm{N}}C_2 + A_6^{\mathrm{N}}(B_2 + 2C_2(wy-1))\right]\frac{\cos wx}{w}\,\mathrm{d}w,$$
$$s_{22}^{\mathrm{NII}}(x,y) = \sigma_{yy}^{\mathrm{NII}} = -\int_0^\infty \mathrm{e}^{-wy}(A_5^{\mathrm{N}} + A_6^{\mathrm{N}}wy)\cos wx\,\mathrm{d}w, \tag{2.20}$$
$$s_{12}^{\mathrm{NII}}(x,y) = \sigma_{xy}^{\mathrm{NII}} = -\int_0^\infty \mathrm{e}^{-wy}\left[A_5^{\mathrm{N}} + A_6^{\mathrm{N}}(wy-1)\right]\sin wx\,\mathrm{d}x,$$
$$s_{11}^{\mathrm{NII}}(x,y) = \sigma_{xx}^{\mathrm{NII}} = \int_0^\infty \mathrm{e}^{-wy}\left[A_5^{\mathrm{N}} + A_6^{\mathrm{N}}(wy-2)\right]\cos wx\,\mathrm{d}w.$$

Here, superscript N refers to normal load. Expressions for A_i^{N} can be found in Appendix A.

2.2. Fundamental solutions of a concentrated load tangential to the half-plane

Similarly to the argument of the previous section, in view of the anti-symmetry of the tangential load (see Fig. 1) and the far-field stress-free condition of substrate, Airy stress functions for the coating and substrate can be sought in the form:

$$F^{\mathrm{I}} = \int_0^\infty [(A_1 + A_2 y)\mathrm{e}^{-wy} + (A_3 + A_4 y)\mathrm{e}^{wy}] \sin wx \, \mathrm{d}w, \tag{2.21}$$

$$F^{\mathrm{II}} = \int_0^\infty (A_5 + A_6 y)\mathrm{e}^{-wy} \sin wx \, \mathrm{d}w, \tag{2.22}$$

where A_i are functions of w to be determined. Using the procedure of Section 2.1 gives

$$\begin{aligned} q_1^{\mathrm{I}} &= \frac{1}{2}\int_0^\infty (A_2 \mathrm{e}^{-wy} - A_4 \mathrm{e}^{wy}) \cos wx \, \mathrm{d}w, \\ q_2^{\mathrm{I}} &= \frac{1}{2}\int_0^\infty (A_2 \mathrm{e}^{-wy} + A_4 \mathrm{e}^{wy}) \sin wx \, \mathrm{d}w, \end{aligned} \tag{2.23}$$

$$\begin{aligned} q_1^{\mathrm{II}} &= \frac{1}{2}\int_0^\infty A_6 \mathrm{e}^{-wy} \cos wx \, \mathrm{d}w, \\ q_2^{\mathrm{II}} &= \frac{1}{2}\int_0^\infty A_6 \mathrm{e}^{-wy} \sin wx \, \mathrm{d}w. \end{aligned} \tag{2.24}$$

Substituting Eqs. (2.21)–(2.24) into Eq. (2.12), the general solutions for the elastic fields of the coating and substrate are obtained. The expressions are not written out explicitly here. From the boundary conditions at the coating surface

$$\begin{aligned} \sigma_{yy}^{\mathrm{I}} &= 0, \\ \sigma_{xy}^{\mathrm{I}} &= -\delta(x) \qquad (y = 0) \end{aligned} \tag{2.25}$$

and the continuity conditions of stresses and displacements along the interface equation (2.17), the coefficients A_i can be specified, as given in Appendix A.

Inserting Eqs. (2.21)–(2.24) into Eq. (2.12), the stress–displacement fields of the coating and the substrate are given. These so-called fundamental solutions for the stresses and displacements can be written as

$$\begin{aligned} d_1^{\mathrm{TI}}(x,y) &= u^{\mathrm{TI}} = -\frac{1}{2}\int_0^\infty \mathrm{e}^{-wy} D_1^{\mathrm{TI}} \frac{\cos wx}{w} \mathrm{d}w, \\ d_2^{\mathrm{TI}}(x,y) &= v^{\mathrm{TI}} = \frac{1}{2}\int_0^\infty \mathrm{e}^{-wy} D_2^{\mathrm{TI}} \frac{\sin wx}{w} \mathrm{d}w, \\ s_{22}^{\mathrm{TI}}(x,y) &= \sigma_{yy}^{\mathrm{TI}} = -\int_0^\infty \left[\mathrm{e}^{-wy}(A_1^{\mathrm{T}} + A_2^{\mathrm{T}} wy) + \mathrm{e}^{wy}(A_3^{\mathrm{T}} + A_4^{\mathrm{T}} wy)\right] \sin wx \, \mathrm{d}w, \\ s_{12}^{\mathrm{TI}}(x,y) &= \sigma_{xy}^{\mathrm{TI}} = -\int_0^\infty \mathrm{e}^{-wy}\left[A_2^{\mathrm{T}}(1 - wy) - (A_1^{\mathrm{T}} - A_3^{\mathrm{T}} \mathrm{e}^{2wy}) + A_4^{\mathrm{T}} \mathrm{e}^{2wy}(1 + wy)\right] \cos wx \, \mathrm{d}w, \\ s_{11}^{\mathrm{TI}}(x,y) &= \sigma_{xx}^{\mathrm{TI}} = \int_0^\infty \mathrm{e}^{-wy}\left[(A_1^{\mathrm{T}} + A_3^{\mathrm{T}} \mathrm{e}^{2wy}) + A_2^{\mathrm{T}}(wy - 2) + A_4^{\mathrm{T}} \mathrm{e}^{2wy}(2 + wy)\right] \sin wx \, \mathrm{d}w, \end{aligned} \tag{2.26}$$

where

$$D_1^{\mathrm{TI}} = 2C_1(A_1^{\mathrm{T}} + A_3^{\mathrm{T}} \mathrm{e}^{2wy}) - A_2^{\mathrm{T}}(B_1 - 2C_1 wy) + A_4^{\mathrm{T}} \mathrm{e}^{2wy}(B_1 + 2C_1 wy),$$

$$
\begin{aligned}
D_2^{\mathrm{TI}}(x,y) &= 2C_1(A_1^{\mathrm{T}} - A_3^{\mathrm{T}}\mathrm{e}^{2wy}) + A_2^{\mathrm{T}}(B_1 + 2C_1(wy-1)) + A_4^{\mathrm{T}}\mathrm{e}^{2wy}(B_1 - 2C_1(wy+1)),\\
d_1^{\mathrm{TII}}(x,y) &= u^{\mathrm{TII}} = \frac{1}{2}\int_0^\infty \mathrm{e}^{-wy}\left(A_6^{\mathrm{T}}B_2 - 2A_5^{\mathrm{T}}C_2 - 2A_6^{\mathrm{T}}C_2wy\right)\frac{\cos wx}{w}\,\mathrm{d}w,\\
d_2^{\mathrm{TII}}(x,y) &= v^{\mathrm{TII}} = \frac{1}{2}\int_0^\infty \mathrm{e}^{-wy}\left[2A_5^{\mathrm{T}}C_2 + A_6^{\mathrm{T}}(B_2 + 2C_2(wy-1))\right]\frac{\sin wx}{w}\,\mathrm{d}w,\\
s_{22}^{\mathrm{TII}}(x,y) &= \sigma_{yy}^{\mathrm{TII}} = -\int_0^\infty \mathrm{e}^{-wy}(A_5^{\mathrm{T}} + A_6^{\mathrm{T}}wy)\sin wx\,\mathrm{d}w, \qquad (2.27)\\
s_{12}^{\mathrm{TII}}(x,y) &= \sigma_{xy}^{\mathrm{TII}} = \int_0^\infty \mathrm{e}^{-wy}\left[A_5^{\mathrm{T}} + A_6^{\mathrm{T}}(wy-1)\right]\cos wx\,\mathrm{d}w,\\
s_{11}^{\mathrm{TII}}(x,y) &= \sigma_{xx}^{\mathrm{TII}} = \int_0^\infty \mathrm{e}^{-wy}\left[A_5^{\mathrm{T}} + A_6^{\mathrm{T}}(wy-2)\right]\sin wx\,\mathrm{d}w,
\end{aligned}
$$

here superscript T refers to the tangential load. The expressions for A_i^{T} can be found in Appendix A.

If the coating material is identical to that of the substrate, or coating thickness approaches infinity, solutions (2.19), (2.21), (2.26) and (2.27) reduce to the classical Flamant's solutions (Hills et al., 1993; Gladwell, 1980).

3. Formulation of singular integral equations of contact of coating–substrate system by fundamental solutions (influence functions)

We now seek Green's functions (GF) for the contact mechanics of coated systems, which provide building blocks used to construct other solutions.

3.1. Influence functions

Putting $y = 0$ in Eq. (2.19) the coating surface displacements are obtained as

$$
\begin{aligned}
u^{\mathrm{NI}} &= \frac{1}{2}\int_0^\infty \left[-A_2^{\mathrm{N}}B_1 + 2A_1^{\mathrm{N}}C_1 + A_4^{\mathrm{N}}B_1 + 2A_3^{\mathrm{N}}C_1\right]\frac{\sin wx}{w}\,\mathrm{d}w,\\
v^{\mathrm{NI}} &= \frac{1}{2}\int_0^\infty \left[A_2^{\mathrm{N}}B_1 + 2A_1^{\mathrm{N}}C_1 - 2A_2^{\mathrm{N}}C_1 + A_4^{\mathrm{N}}B_1 - 2A_3^{\mathrm{N}}C_1 - 2A_4^{\mathrm{N}}C_1\right]\frac{\cos wx}{w}\,\mathrm{d}w
\end{aligned} \qquad (3.1)
$$

and their derivatives with respect to x as

$$
\begin{aligned}
\frac{\partial u^{\mathrm{NI}}}{\partial x} &= \frac{1}{2}\int_0^\infty \left[-A_2^{\mathrm{N}}B_1 + 2A_1^{\mathrm{N}}C_1 + A_4^{\mathrm{N}}B_1 + 2A_3^{\mathrm{N}}C_1\right]\cos wx\,\mathrm{d}w,\\
\frac{\partial v^{\mathrm{NI}}}{\partial x} &= \frac{1}{2}\int_0^\infty \left[-A_2^{\mathrm{N}}B_1 - 2A_1^{\mathrm{N}}C_1 + 2A_2^{\mathrm{N}}C_1 - A_4^{\mathrm{N}}B_1 + 2A_3^{\mathrm{N}}C_1 + 2A_4^{\mathrm{N}}C_1\right]\sin wx\,\mathrm{d}w.
\end{aligned} \qquad (3.2)
$$

By letting

$$
\begin{aligned}
&\frac{1}{2}\left[2A_1^{\mathrm{N}}C_1 + 2A_3^{\mathrm{N}}C_1 - A_2^{\mathrm{N}}B_1 + A_4^{\mathrm{N}}B_1\right] = G_1\{1 + R_1\},\\
&\frac{1}{2}\left[2A_2^{\mathrm{N}}C_1 - 2A_1^{\mathrm{N}}C_1 + 2A_3^{\mathrm{N}}C_1 + 2A_4^{\mathrm{N}}C_1 - A_2^{\mathrm{N}}B_1 - A_4^{\mathrm{N}}B_1\right] = G_2\{1 + R_2\},
\end{aligned} \qquad (3.3)
$$

where $G_2 = \frac{2(\nu_1^2-1)}{E_1\pi}$, $G_1 = \frac{(1+\nu_1)(2\nu_1-1)}{E_1\pi}$, Eq. (3.2) can be rewritten as

$$\begin{aligned}\frac{\partial u^{\mathrm{NI}}}{\partial x} &= G_1\int_0^\infty [1+R_1(hw)]\cos wx\,\mathrm{d}w = G_1\left[\pi\delta(x)+\int_0^\infty R_1(hw)\cos wx\,\mathrm{d}x\right],\\ \frac{\partial v^{\mathrm{NI}}}{\partial x} &= G_2\int_0^\infty [1+R_2(hw)]\sin wx\,\mathrm{d}w = G_2\left[\frac{1}{x}+\int_0^\infty R_2(hw)\sin wx\,\mathrm{d}w\right],\end{aligned} \tag{3.4}$$

where the relations

$$\delta(x) = \frac{1}{\pi}\int_0^\infty \cos wx\,\mathrm{d}w, \qquad \frac{1}{x} = \int_0^\infty \sin wx\,\mathrm{d}w \tag{3.5}$$

are used, and

$$R_1(W) = \frac{R_{11}(W)}{R_{12}(W)} - 1,$$

$$R_{11}(W) = R_{111}(W) + R_{112}(W),$$

$$R_{111}(W) = -(B_1-2C_1)(B_2+C)(B_1-C) - (B_1-2C_1)C(B_1-B_2-C)\,\mathrm{e}^{-4W},$$

$$\begin{aligned}R_{112}(W) = \mathrm{e}^{-2W}\{&2(B_1-2B_2-2C_1)(B_1-C_1)C_1 + 2C_2(B_1B_2+4B_1C_1-2B_2C_1-4C_1^2\\ &-B_1C_2+2C_1C_2)+8C_1C(B_2+C)W^2\},\end{aligned}$$

$$\begin{aligned}R_{12}(W) = 2G_1\pi\{&(B_2+C)(B_1-C)+C(B_1-B_2-C)\mathrm{e}^{-4W}\\ &+\mathrm{e}^{-2W}[B_1(B_1-B_2-2C)+2C(B_2+C)(1+2W^2)]\},\end{aligned}$$

$$R_2(W) = -1 - \frac{B_1(B_1-C)(B_2+C) - B_1(B_1-B_2-C)C\mathrm{e}^{-4W} - 4B_1C\mathrm{e}^{-2W}(B_2+C)W}{2G_2\pi\{(B_1-C)(B_2+C)+(B_1-B_2-C)C\mathrm{e}^{-4W}+\mathrm{e}^{-2W}[B_1^2-B_1(B_2+2C)+2C(B_2+C)(1+2W^2)]\}}$$

where $W = wh$.

Eqs. (3.4) are Green's functions, or influence functions due to the normal unity load.

Similarly, putting $y = 0$ in Eq. (2.26) one obtains the coating surface displacements as

$$\begin{aligned}u^{\mathrm{TI}} &= -\frac{1}{2}\int_0^\infty \left[2C_1(A_1^{\mathrm{T}}+A_3^{\mathrm{T}}) - A_2^{\mathrm{T}}B_1 + A_4^{\mathrm{T}}B_1\right]\frac{\cos wx}{w}\mathrm{d}w,\\ v^{\mathrm{TI}} &= \frac{1}{2}\int_0^\infty \left[2C_1(A_1^{\mathrm{T}}-A_3^{\mathrm{T}}) + A_2^{\mathrm{T}}(B_1-2C_1) + A_4^{\mathrm{T}}(B_1-2C_1)\right]\frac{\sin wx}{w}\mathrm{d}w\end{aligned} \tag{3.6}$$

and their divertive with respect to x as

$$\begin{aligned}\frac{\partial u^{\mathrm{TI}}}{\partial x} &= \frac{1}{2}\int_0^\infty \left[2C_1(A_1^{\mathrm{T}}+A_3^{\mathrm{T}}) - A_2^{\mathrm{T}}B_1 + A_4^{\mathrm{T}}B_1\right]\sin wx\,\mathrm{d}w,\\ \frac{\partial v^{\mathrm{TI}}}{\partial x} &= \frac{1}{2}\int_0^\infty \left[2C_1(A_1^{\mathrm{T}}-A_3^{\mathrm{T}}) + (A_2^{\mathrm{T}}+A_4^{\mathrm{T}})(B_1-2C_1)\right]\cos wx\,\mathrm{d}w.\end{aligned} \tag{3.7}$$

As in the previous derivation, let

$$\begin{aligned}&\frac{1}{2}\left[-A_2^{\mathrm{T}}B_1 + 2A_1^{\mathrm{T}}C_1 + A_4^{\mathrm{T}}B_1 + 2A_3^{\mathrm{T}}C_1\right] = G_3(1+R_3),\\ &\frac{1}{2}\left[A_2^{\mathrm{T}}B_1 + 2A_1^{\mathrm{T}}C_1 - 2A_2^{\mathrm{T}}C_1 + A_4^{\mathrm{T}}B_1 - 2A_3^{\mathrm{T}}C_1 - 2A_4^{\mathrm{T}}C_1\right] = G_4(1+R_4),\end{aligned} \tag{3.8}$$

where $G_3 = G_2$, $G_4 = -G_1$. Then Eq. (3.7) is transformed into

$$\begin{aligned}\frac{\partial u^{\mathrm{TI}}}{\partial x} &= G_3 \int_0^{\infty} [1 + R_3(hw)] \sin wx \,\mathrm{d}w = G_3 \left[\frac{1}{x} + \int_0^{\infty} R_3(hw) \sin wx \,\mathrm{d}w \right], \\ \frac{\partial v^{\mathrm{TI}}}{\partial x} &= G_4 \int_0^{\infty} [1 + R_4(hw)] \cos wx \,\mathrm{d}w = G_4 \left[\pi\delta(x) + \int_0^{\infty} R_4(hw) \cos wx \,\mathrm{d}w \right],\end{aligned} \tag{3.9}$$

where

$$R_3(W) = -1 - \frac{B_1(B_1 - C)(B_2 + C) - B_1(B_1 - B_2 - C)C\mathrm{e}^{-4W} + 4B_1 C\mathrm{e}^{-2W}(B_2 + C)W}{2G_3\pi\{(B_1 - C)(B_2 + C) + (B_1 - B_2 - C)C\mathrm{e}^{-4W} + \mathrm{e}^{-2W}[B_1^2 - B_1(B_2 + 2C) + 2C(B_2 + C)(1 + 2W^2)]\}}$$

and

$$R_4(W) = -1 + \frac{R_{41}(W)}{2\pi G_4 R_{42}(W)},$$

$$R_{41}(W) = R_{411}(W) + R_{412}(W),$$

$$R_{411}(W) = (B_1 - 2C_1)(B_1 - C)(B_2 + C) + (B_1 - 2C_1)(B_1 - B_2 - C)C\mathrm{e}^{-4W},$$

$$R_{412}(W) = -2\mathrm{e}^{-2W}\{B_1^2 C_1 - B_1[B_2(C_1 + C) + C(2C_1 + C)] + 2C_1 C(B_2 + C)(1 + 2W^2)\},$$

$$R_{42}(W) = R_{421}(W) + R_{422}(W),$$

$$R_{421}(W) = (B_1 - C)(B_2 + C) + (B_1 - B_2 - C)C\mathrm{e}^{-4W},$$

$$R_{422}(W) = \mathrm{e}^{-2W}[B_1^2 - B_1(B_2 + 2C) + 2C(B_2 + C)(1 + 2W^2)].$$

Eqs. (3.9) are Green's functions or influence functions due to the tangential unity load. It must be noted that $\lim_{W\to\infty} R_i(W) = 0$ $(i = 1$–$4)$.

3.2. Formulation of fundamental integral equations for contact of coated system

Using Eqs. (3.4) and (3.9), the displacement derivatives of coating surface loaded by a distribution of frictions can be written by the Green's function method as follows:

$$\begin{aligned}\frac{\partial u_1(x)}{\partial x} &= \int \frac{\partial u^{\mathrm{NI}}(x-t)}{\partial t} p(t)\,\mathrm{d}t + \int \frac{\partial u^{\mathrm{TI}}(x-t)}{\partial t} q(t)\,\mathrm{d}t, \\ \frac{\partial v_1(x)}{\partial x} &= \int \frac{\partial v^{\mathrm{NI}}(x-t)}{\partial t} p(t)\,\mathrm{d}t + \int \frac{\partial v^{\mathrm{TI}}(x-t)}{\partial t} q(t)\,\mathrm{d}t,\end{aligned} \tag{3.10}$$

where $p(t)$ and $q(t)$ are, respectively, unknown continuous normal and tangential tractions, and u_1, v_1 are normal and tangential displacements of coating surface due to combination loads of $p(t)$ and $q(t)$, and t is the dummy integration variable whose range of variation is the contact zone.

Inserting (3.4) and (3.9) into (3.10) leads to

$$\begin{aligned}\frac{\partial u_1(x)}{\partial x} = &\int G_1 \left[\pi\delta(x-t) + \int_0^{\infty} R_1(hw) \cos w(x-t)\,\mathrm{d}x \right] p(t)\,\mathrm{d}t \\ &+ \int G_3 \left[\frac{1}{x-t} + \int_0^{\infty} R_3(hw) \sin w(x-t)\,\mathrm{d}w \right] q(t)\,\mathrm{d}t, \\ \frac{\partial v_1(x)}{\partial x} = &\int G_2 \left[\frac{1}{x-t} + \int_0^{\infty} R_2(hw) \sin w(x-t)\,\mathrm{d}w \right] p(t)\,\mathrm{d}t \\ &+ \int G_4 \left[\pi\delta(x-t) + \int_0^{\infty} R_4(hw) \cos w(x-t)\,\mathrm{d}w \right] q(t)\,\mathrm{d}t\end{aligned} \tag{3.11}$$

or

$$\begin{aligned}\frac{\partial u_1(x)}{\partial x} &= G_1\pi p(x) + G_2\int\frac{q(t)}{x-t}\mathrm{d}t + \int G_1\left[\int_0^\infty R_1(hw)\cos w(x-t)\,\mathrm{d}t\right]p(t)\,\mathrm{d}t \\ &\quad + \int G_2\left[\int_0^\infty R_3(hw)\sin w(x-t)\,\mathrm{d}w\right]q(t)\,\mathrm{d}t, \\ \frac{\partial v_1(x)}{\partial x} &= G_2\int\frac{p(t)}{x-t}\mathrm{d}t - G_1\pi q(x) + G_2\int\left[\int_0^\infty R_2(hw)\sin w(x-t)\,\mathrm{d}w\right]p(t)\,\mathrm{d}t \\ &\quad - G_1\int\left[\int_0^\infty R_4(hw)\cos w(x-t)\,\mathrm{d}w\right]q(t)\,\mathrm{d}t\end{aligned} \tag{3.12}$$

by virtue of $G_3 = G_2$, $G_4 = -G_1$.

Now, let us turn to the surface deformation of an elastic indenter. Consider the uncoated case first, which can be easily obtained by setting the materials of coating and substrate to be identical in (3.12) and denoting Poisson's ratio and Young's modulus of the indenter by v_3 and E_3. Consequently, the deformation of an elastic indenter can be presented as

$$\begin{aligned}\frac{\partial u_3(x)}{\partial x} &= G_5\pi p(x) + G_6\int\frac{q(t)}{x-t}\mathrm{d}t, \\ \frac{\partial v_3(x)}{\partial x} &= G_6\int\frac{p(t)}{x-t}\mathrm{d}t - G_5\pi q(x),\end{aligned} \tag{3.13}$$

where $G_5 = \frac{(1+v_3)(2v_3-1)}{E_3\pi}$, $G_6 = \frac{2(v_3^2-1)}{E_3\pi}$.

In the coordinates of coating and substrate as shown in Fig. 2, Eq. (3.13) in the global coordinates should be rewritten as

$$\begin{aligned}\frac{\partial u_3(x)}{\partial x} &= G_5\pi p(x) - G_6\int\frac{q(t)}{x-t}\mathrm{d}t, \\ \frac{\partial v_3(x)}{\partial x} &= -G_6\int\frac{p(t)}{x-t}\mathrm{d}t - G_5\pi q(x).\end{aligned} \tag{3.14}$$

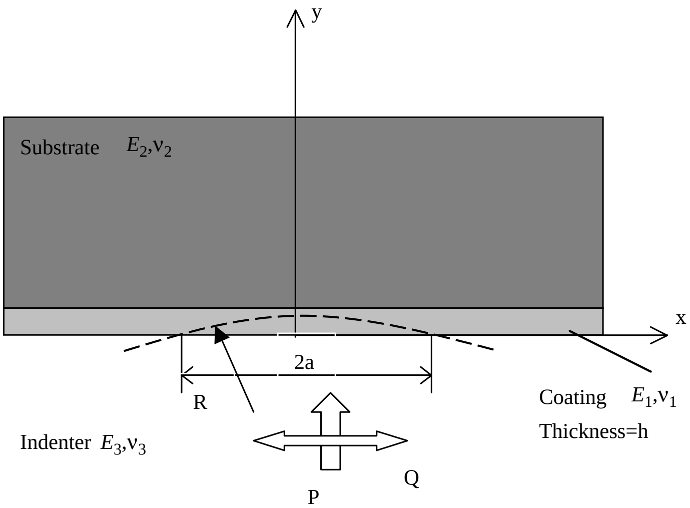

Fig. 2. Contact of two gently curved bodies.

Following Hertz, it will be assumed that the contact width is small compared with the radii of the curvature of the contacting bodies. Thus, each may be replaced by a half-plane, and the value of the relative displacements $g_x = u_1 - u_3$, $g_y = v_1 - v_3$ can be expressed as

$$\begin{aligned}\frac{\partial g_x(x)}{\partial x} &= (G_1 - G_5)\pi p(x) + (G_2 + G_6)\int \frac{q(t)}{x-t}\mathrm{d}t + \int G_1\left[\int_0^\infty R_1(hw)\cos w(x-t)\,\mathrm{d}t\right]p(t)\,\mathrm{d}t \\ &\quad + \int G_2\left[\int_0^\infty R_3(hw)\sin w(x-t)\,\mathrm{d}w\right]q(t)\,\mathrm{d}t, \\ \frac{\partial g_y(x)}{\partial x} &= (G_2 + G_6)\int \frac{p(t)}{x-t}\mathrm{d}t - (G_1 - G_5)\pi q(x) + G_2\int\left[\int_0^\infty R_2(hw)\sin w(x-t)\,\mathrm{d}w\right]p(t)\,\mathrm{d}t \\ &\quad - G_1\int\left[\int_0^\infty R_4(hw)\cos w(x-t)\,\mathrm{d}w\right]q(t)\,\mathrm{d}t,\end{aligned} \tag{3.15}$$

which also can be re-expressed in a standard form as

$$\begin{aligned}\frac{1}{A}\frac{\partial g_x(x)}{\partial x} &= \beta p(x) + \frac{1}{\pi}\int \frac{q(t)}{x-t}\mathrm{d}t + \beta_1\frac{1}{\pi}\int\left[\int_0^\infty R_1(hw)\cos w(x-t)\,\mathrm{d}w\right]p(t)\,\mathrm{d}t \\ &\quad + \beta_2\frac{1}{\pi}\int\left[\int_0^\infty R_3(hw)\sin w(x-t)\,\mathrm{d}w\right]q(t)\,\mathrm{d}t, \\ \frac{1}{A}\frac{\partial g_y(x)}{\partial x} &= \frac{1}{\pi}\int \frac{p(t)}{x-t}\mathrm{d}t - \beta q(x) + \beta_2\frac{1}{\pi}\int\left[\int_0^\infty R_2(hw)\sin w(x-t)\,\mathrm{d}w\right]p(t)\,\mathrm{d}t \\ &\quad - \beta_1\frac{1}{\pi}\int\left[\int_0^\infty R_4(hw)\cos w(x-t)\,\mathrm{d}w\right]q(t)\,\mathrm{d}t,\end{aligned} \tag{3.16}$$

where

$$A = \frac{2(v_1^2 - 1)}{E_1} + \frac{2(v_3^2 - 1)}{E_3}, \qquad \beta = \frac{(1 + v_1)(2v_1 - 1)/E_1 - (1 + v_3)(2v_3 - 1)/E_3}{2(v_1^2 - 1)/E_1 + 2(v_3^2 - 1)/E_3},$$

$$\beta_1 = \frac{1}{A}\frac{(1 + v_1)(2v_1 - 1)}{E_1}, \qquad \beta_2 = \frac{1}{A}\frac{2(v_1^2 - 1)}{E_1}.$$

Eqs. (3.16) are the fundamental equations for the contact between an elastic indenter and a coated system. It is understood that the integrals are carried out over the entire contact zone in each case. Clearly, the normal traction and tangential traction are coupled in Eq. (3.16). Only if all materials of coating, substrate and indenter are identical, then Eq. (3.16) can be decoupled. This is different from the contact situation in the uncoated problem. In the uncoated case, when the substrate material is identical to the indenter material, the contact equations are decoupled. In addition, we ensure equilibrium with the external forces P, Q by requiring

$$P = \int p(t)\,\mathrm{d}t, \tag{3.17}$$

$$Q = \int q(t)\,\mathrm{d}t. \tag{3.18}$$

4. Special cases of contact of coating–substrate system

From the fundamental equations (3.16), a series of special cases of the governing equations can be obtained by selecting appropriate material parameters. Below, we present some frequently encountered cases. Some of them correspond to practical contact situations studied by other models and methods, such as the FEM (Tian and Saka, 1991; Anderson and Collins, 1995), BIM (Gupta and Walowit, 1974; Elsharkawy, 1999) and a hybrid method (Bentall and Johnson, 1986; Nowell and Hills, 1988).

4.1. Rigid indenter

When the indenter is rigid, Eq. (3.16) can be written into the form

$$\begin{aligned}\frac{1}{A}\frac{\partial g_x(x)}{\partial x} &= \beta p(x) + \frac{1}{\pi}\int\frac{q(t)}{x-t}\mathrm{d}t + \frac{\beta}{\pi}\int\left[\int_0^\infty R_1(hw)\cos w(x-t)\,\mathrm{d}t\right]p(t)\,\mathrm{d}t \\ &\quad + \frac{1}{\pi}\int\left[\int_0^\infty R_3(hw)\sin w(x-t)\,\mathrm{d}w\right]q(t)\,\mathrm{d}t, \\ \frac{1}{A}\frac{\partial g_y(x)}{\partial x} &= \frac{1}{\pi}\int\frac{p(t)}{x-t}\mathrm{d}t - \beta q(x) + \frac{1}{\pi}\int\left[\int_0^\infty R_2(hw)\sin w(x-t)\,\mathrm{d}w\right]p(t)\,\mathrm{d}t \\ &\quad - \frac{\beta}{\pi}\int\left[\int_0^\infty R_4(hw)\cos w(x-t)\,\mathrm{d}w\right]q(t)\,\mathrm{d}t\end{aligned} \tag{4.1}$$

with $A = \frac{2(v_1^2-1)}{E_1}$, $\beta = \frac{(1-2v_1)}{2(1-v_1)}$. Here, it should be pointed out that the conditions of validity of Eq. (4.1) are less strict than those for Eq. (3.16): the radius of curvature of the coating surface must be much larger than the contact width, but there is no restriction for curvature of the rigid indenter. This case can be often used when indenter is much stiffer than coating. Numerical solution of Eq. (4.1), particularly when partial slip occurs, requires introducing some assumptions, e.g. Amontons (or Coulomb) friction law, or the Goodman assumption, similarly to the uncoated case (Hills et al., 1993).

4.2. Fully sliding frictional case

Considering the relative displacement of the contact surfaces in tangential direction, contact problems can be classified into three cases:

- The full stick problem: once a point at the surface of indenter comes into contact with a corresponding coating surface point, their relative displacement in the tangential direction is fixed at a constant value during subsequent increase of the contact load.
- Partial slip: during increase of external loading at some points on the contact surface shear tractions reach a limiting value, and slip (change in the relative displacement of contacting points) takes place.
- Full slip: the entire contact surface is under sliding conditions.

In practice, full stick seldom happens in the case of incomplete contacts. Partial slip problem presents a numerically complex problem even for the uncoated case (Nowell and Hills, 1988), and is even more difficult in the coated case. In this paper we consider the case of full slip of a frictional contact. We assume that the relation $q(t) = fp(t)$ holds everywhere, where f is the friction coefficient. Eq. (3.16) then can be reduced to a single Fredholm integral equation of the second kind as

$$\begin{aligned}\frac{1}{A}\frac{\partial g_y(x)}{\partial x} &= \frac{1}{\pi}\int\frac{p(t)}{x-t}\mathrm{d}t - f\beta p(x) + \frac{\beta_2}{\pi}\int\left[\int_0^\infty R_2(hw)\sin w(x-t)\,\mathrm{d}w\right]p(t)\,\mathrm{d}t \\ &\quad - \frac{f\beta_1}{\pi}\int\left[\int_0^\infty R_4(hw)\cos w(x-t)\,\mathrm{d}w\right]p(t)\,\mathrm{d}t.\end{aligned} \tag{4.2}$$

Moreover, if the indenter is rigid, Eq. (4.2) can be simplified to

$$\frac{1}{A}\frac{\partial g_y(x)}{\partial x} = -\beta f p(x) + \frac{1}{\pi}\int \frac{p(t)}{x-t}\,\mathrm{d}t + \frac{1}{\pi}\int k(x,t)p(t)\,\mathrm{d}t, \tag{4.3}$$

where

$$k(x,t) = \int_0^\infty [R_2(hw)\sin w(x-t) - f\beta R_4(hw)\cos w(x-t)]\,\mathrm{d}w.$$

A set of powerful methods have been proposed to solve this kind of singular Fredholm integral (Erdogan et al., 1973; Ma and Korsunsky, 2002).

4.3. Frictionless case

If the friction coefficient is equal to zero, Eq. (4.2) reduces to the Fredholm integral equation of the first kind

$$\frac{1}{A}\frac{\partial g_y(x)}{\partial x} = \frac{1}{\pi}\int \frac{p(t)}{x-t}\,\mathrm{d}t + \beta_2\frac{1}{\pi}\int \left[\int_0^\infty R_2(hw)\sin w(x-t)\,\mathrm{d}w\right]p(t)\,\mathrm{d}t. \tag{4.4}$$

Furthermore, if the indenter is rigid, Eq. (4.4) degenerates to

$$\frac{1}{A}\frac{\partial g_y(x)}{\partial x} = \frac{1}{\pi}\int \frac{p(t)}{x-t}\,\mathrm{d}t + \frac{1}{\pi}\int \left[\int_0^\infty R_2(hw)\sin w(x-t)\,\mathrm{d}w\right]p(t)\,\mathrm{d}t. \tag{4.5}$$

In the literature, frictionless cases have been investigated by many authors (Gupta and Walowit, 1974; and others).

4.4. Thin coating

When the coating material is identical to substrate material, or the thickness of coating $h \to \infty$, Eqs. (3.16) reduce to the equations for the uncoated system obtained by Hills et al. (1993) (Eqs. (2.17) and (2.22)).

$$\begin{aligned}\frac{1}{A}\frac{\partial g_x(x)}{\partial x} &= \beta p(x) + \frac{1}{\pi}\int \frac{q(t)}{x-t}\,\mathrm{d}t,\\ \frac{1}{A}\frac{\partial g_y(x)}{\partial x} &= \frac{1}{\pi}\int \frac{p(t)}{x-t}\,\mathrm{d}t - \beta q(x),\end{aligned} \tag{4.6}$$

where $A = \frac{2(v_1^2-1)}{E_1} + \frac{2(v_3^2-1)}{E_3}$, $\beta = \frac{(1+v_1)(2v_1-1)/E_1-(1+v_3)(2v_3-1)/E_3}{2(v_1^2-1)/E_1+2(v_3^2-1)/E_3}$.

Sometimes, attention is focused on the larger deformation along the coating. It is possible to assume approximately that both substrate and indenter are rigid. The full slip contact equation can be reduced to the Fredholm singular integral of the second kind:

$$\begin{aligned}\frac{1}{A}\frac{\partial g_y(x)}{\partial x} = {} & \frac{1}{\pi}\int \frac{p(t)}{x-t}\,\mathrm{d}t - \beta f q(x) + \frac{1}{\pi}\int \left[\int_0^\infty R_2(hw)\sin w(x-t)\,\mathrm{d}w\right]p(t)\,\mathrm{d}t\\ & -\frac{f\beta}{\pi}\int \left[\int_0^\infty R_4(hw)\cos w(x-t)\,\mathrm{d}w\right]q(t)\,\mathrm{d}t,\end{aligned} \tag{4.7}$$

where $A = \frac{2(v_1^2-1)}{E_1}$, $\beta = \frac{(1-2v_1)}{2(1-v_1)}$, and

$$R_2(W) = -1 - \frac{B_1(B_1 - C_1)C_1(1 - e^{-4W}) - 4B_1C_1^2 e^{-2W} W}{2G_2\pi\{(B_1 - C_1)C_1(1 + e^{-4W}) + e^{-2W}[B_1^2 - 2C_1B_1 + 2C_1^2(1 + 2W^2)]\}},$$

$$R_4(W) = -1 + \frac{(B_1 - 2C_1)(B_1 - C_1)C_1(1 + e^{-4W}) - 2e^{-2W}\left[B_1^2C_1 - 3C_1^2B_1 + 2C_1^3(1 + 2W^2)\right]}{2\pi G_4\{(B_1 - C_1)C_1(1 + e^{-4W}) + e^{-2W}[B_1^2 - 2C_1B_1 + 2C_1^2(1 + 2W^2)]\}}.$$

5. Stress and displacement fields in coating and substrate

Unknown traction distributions $p(x)$ and $q(x)$ must be solved by some suitable numerical methods for the inversion of singular integral equations (e.g., Erdogan et al., 1973; Ma and Korsunsky, 2002). If $p(x)$ and $q(x)$ in Sections 3 and 4 are found, then stress and displacement fields in the coating and substrate obtain respectively as follows:

Coating:

$$\begin{aligned} d_i^{\mathrm{I}}(x,y) &= \int \left[d_i^{\mathrm{NI}}(x-t,y)p(t) + d_i^{\mathrm{TI}}(x-t,y)q(t)\right]\mathrm{d}t, \\ s_{ij}^{\mathrm{I}}(x,y) &= \int \left[s_{ij}^{\mathrm{NI}}(x-t,y)p(t) + s_{ij}^{\mathrm{TI}}(x-t,y)q(t)\right]\mathrm{d}t. \end{aligned} \tag{5.1}$$

Substrate:

$$\begin{aligned} d_i^{\mathrm{II}}(x,y) &= \int \left[d_i^{\mathrm{NII}}(x-t,y)p(t) + d_i^{\mathrm{TII}}(x-t,y)q(t)\right]\mathrm{d}t, \\ s_{ij}^{\mathrm{II}}(x,y) &= \int \left[s_{ij}^{\mathrm{NII}}(x-t,y)p(t) + s_{ij}^{\mathrm{TII}}(x-t,y)q(t)\right]\mathrm{d}t, \end{aligned} \tag{5.2}$$

where $d_i^{\mathrm{NI}}(x,y)$, $d_i^{\mathrm{NII}}(x,y)$, $s_{ij}^{\mathrm{NI}}(x,y)$, $s_{ij}^{\mathrm{NII}}(x,y)$, $d_i^{\mathrm{TI}}(x,y)$, $d_i^{\mathrm{TII}}(x,y)$, $s_{ij}^{\mathrm{TI}}(x,y)$ and $s_{ij}^{\mathrm{TII}}(x,y)$ are, respectively, the displacement and stress kernel functions that can be found in Eqs. (2.19), (2.21), (2.26) and (2.27).

6. Numerical example

Consider the problem of a rigid cylindrical punch, sliding on a coated elastic half-plane as shown in Fig. 2. The problem can be expressed by Eq. (4.3) and solved using the Erdogan method (Erdogan et al., 1973). Without wishing to discuss the details of the numerical implementation we focus here on the variation of the traction distribution along the coating surface due to the introduction of the coating and further with the change in the friction coefficient.

The elastic parameters of the softer coating and the stiffer substrate are as follows:

Indenter: rigid, indenter radius: $R = 5.0 \times 10^{-3}$ m.
Substrate: $E_2 = 1.15 \times 10^{11}$ Pa, $\nu_2 = 0.33$.
Coating: $E_1 = E_2/2$, $\nu_1 = 0.33$, coating thickness: $h = 2 \times 10^{-5}$ m.
Normal load: $P = 15{,}000$ N/m.

Fig. 3 shows the traction profiles along the contact. Further numerical results for the eccentricity and extent of contact are given in Table 1.

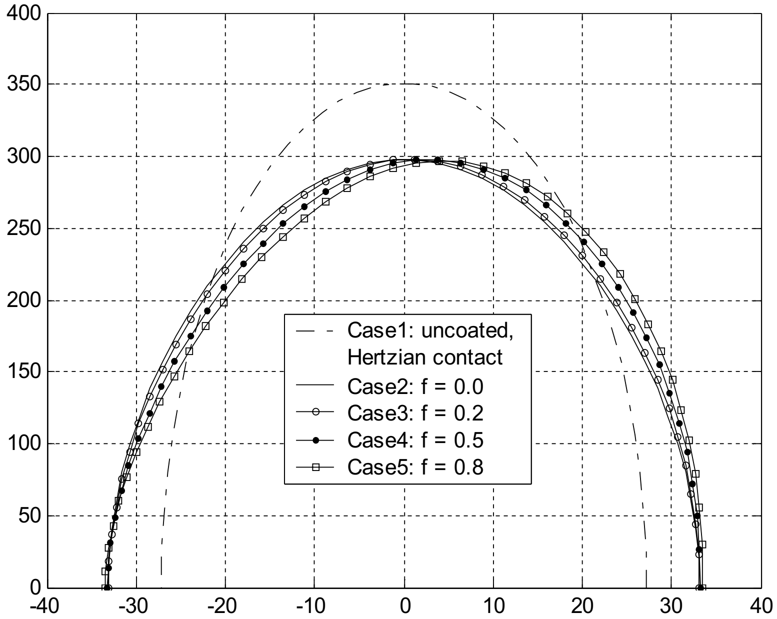

Fig. 3. Normal traction vs. friction coefficient for a rigid Hertzian indenter sliding over a coated half-plane under normal load $P = 15$ kN/m.

Table 1
Variation of contact parameters with the friction coefficient for a rigid Hertzian indenter sliding over a coated half-plane under normal load $P = 15$ kN/m

Friction coefficient, f	Contact half width, a (μm)	Eccentricity parameter, e (μm)	Thickness parameter (h/a)
Case 1: $f = 0.0$ (uncoated)	27.20	0.0	0.0
Case 2: $f = 0.0$	33.10	0.0	0.6
Case 3: $f = 0.2$	33.13	2.6	0.6
Case 4: $f = 0.5$	33.26	6.6	0.6
Case 5: $f = 0.8$	33.49	10.6	0.6

Numerical results for the pressure distribution for the uncoated case are in perfect agreement with the Hertzian formula for the semi width of a two-dimensional contact between a rigid cylindrical punch and an elastic half-plane:

$$a = \sqrt{\frac{4PR(1 - \nu^2)}{\pi E}}. \tag{6.1}$$

In the case of uncoated substrate the above formula gives the value of 27.20 μm, as in Table 1. If, on the other hand, a semi-infinite solid with the elastic properties of the coating were considered, the result would be 38.47 μm. The numerical results obtained for the coated substrate are expected to lie between these two extremes, as confirmed in Table 1.

Compared with the normal traction distribution of the Hertzian contact (i.e., uncoated, frictionless contact, Case 1 in Fig. 3), the tractions for the coated contact are reduced due to the increased extent of the contact. With increasing coefficient of friction the contact semi-width increases, as does the eccentricity. The phenomena associated with the change in the extent of contact are relatively mild in the Hertzian case. However, they are likely to be much more significant in other cases, e.g. that of flat-and-rounded contacts, where significant increase of the normal traction towards the edge of contact is observed. These effects will be treated separately.

7. Concluding remarks

Fundamental solutions for the concentrated normal and tangential forces acting at the surface of a coated half-plane have been obtained.

On the basis of the fundamental solutions for concentrated forces, singular integral equation formulation has been developed for the unknown traction distributions, for the general case of frictional contact between an elastic punch and a coated substrate. This is a general basic framework for the analysis of contact of coated system.

Some special integral equation formulations have been derived corresponding to some typical cases of contacts of coated systems, which are often encountered in practical situations.

Subsequently, full displacement and stress fields of coated system due to arbitrary surface tractions have been derived by Green's function method and presented in simple forms.

Finally, a typical example is considered and the numerical solution given to support the validity of the fundamental equations deduced in this paper.

The conclusions obtained in this paper apply to the contact of a system coated by single layer. For the contact of multi-coated system fundamental solutions can be re-derived using the general procedures given in this paper.

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Appendix A

A.1. A_i and A_i^N due to concentrated normal load

$$A_1 = \frac{A_1^{\mathrm{N}}}{w^2} = \frac{2B_1B_2 + 2(B_1 - B_2 - C)C + \mathrm{e}^{-2hw}\left[B_1^2 - B_1B_2 - 2(B_1 - B_2 - C)C + 4C(B_2 + C)hw(hw - 1)\right]}{2\pi w^2 D},$$

$$A_2 = \frac{A_2^{\mathrm{N}}}{w} = -\frac{(C - B_1)(B_2 + C) + (B_2 + C)\mathrm{e}^{-2hw}C(2hw - 1)}{\pi w D},$$

$$A_3 = \frac{A_3^{\mathrm{N}}}{w^2} = \frac{2(B_1 - B_2 - C)C\mathrm{e}^{-4hw} + \mathrm{e}^{-2hw}\left[B_1^2 - B_1B_2 - 2(B_1 - B_2 - C)C + 4C(B_2 + C)hw(hw + 1)\right]}{2\pi w^2 D},$$

$$A_4 = \frac{A_4^{\mathrm{N}}}{w} = -\frac{(B_1 - B_2 - C)C\,\mathrm{e}^{-4hw} + \mathrm{e}^{-2hw}C(B_2 + C)(1 + 2hw)}{\pi w D},$$

$$A_5 = \frac{A_5^{\mathrm{N}}}{w^2} = -\frac{B_1[-B_1 - B_2 + 2(B_1 - B_2 - 2C)hw] + B_1\,\mathrm{e}^{-2hw}[-B_1 + B_2 + 4Chw(1 - hw)]}{2\pi w^2 D},$$

$$A_6 = \frac{A_6^{\mathrm{N}}}{w} = \frac{B_1(B_1 - C) + B_1\,\mathrm{e}^{-2hw}C(1 - 2hw)}{\pi w D},$$

$$D = \left(B_1B_2 + (B_1 - B_2 - C)C(1 + \mathrm{e}^{-4hw}) + \mathrm{e}^{-2hw}\left[B_1^2 - B_1B_2 - 2(B_1 - B_2 - C)C + 4C(B_2 + C)h^2w^2\right]\right).$$

A.2. A_i and A_i^N due to concentrated tangential load

$$A_1 = \frac{A_1^{\mathrm{T}}}{w^2} = -\frac{\mathrm{e}^{-2hw}[B_1(B_1 - B_2 - 2C) + 4C(B_2 + C)h^2w^2]}{2\pi w^2 D},$$

$$A_2 = \frac{A_2^{\mathrm{T}}}{w} = -\frac{(C - B_1)(B_2 + C) - (B_2 + C)\mathrm{e}^{-2hw}C(1 + 2hw)}{\pi w D},$$

$$A_3 = \frac{A_3^{\mathrm{T}}}{w^2} = \frac{\mathrm{e}^{-2hw}[B_1(B_1 - B_2 - 2C) + 4C(B_2 + C)h^2w^2]}{2\pi w^2 D},$$

$$A_4 = \frac{A_4^{\mathrm{T}}}{w} = \frac{C(B_1 - B_2 - C)\mathrm{e}^{-4hw} + \mathrm{e}^{-2hw}C(B_2 + C)(1 - 2hw)}{\pi w D},$$

$$A_5 = \frac{A_5^{\mathrm{T}}}{w^2} = -\frac{B_1(B_1 - B_2 - 2C)(2hw - 1) + B_1\mathrm{e}^{-2hw}[B_1 - B_2 - 2C + 4Ch^2w^2]}{2\pi w^2 D},$$

$$A_6 = \frac{A_6^{\mathrm{T}}}{w} = \frac{B_1(B_1 - C) + B_1\,\mathrm{e}^{-2hw}C(1 + 2hw)}{\pi w D},$$

$$D = \left(B_1B_2 + (B_1 - B_2 - C)C(1 + \mathrm{e}^{-4hw}) + \mathrm{e}^{-2hw}\left[B_1^2 - B_1B_2 - 2(B_1 - B_2 - C)C + 4C(B_2 + C)h^2w^2\right]\right).$$

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